

On Two Strong Converse Theorems for Discrete Memoryless Channels*

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SUMMARY In 1973, Arimoto proved the strong converse theorem for the discrete memoryless channels stating that when transmission rate R is above channel capacity C , the error probability of decoding goes to one as the block length n of code word tends to infinity. He proved the theorem by deriving the exponent function of error probability of correct decoding that is positive if and only if $R > C$. Subsequently, in 1979, Dueck and Körner determined the optimal exponent of correct decoding. Arimoto's bound has been said to be equal to the bound of Dueck and Körner. However its rigorous proof has not been presented so far. In this paper we give a rigorous proof of the equivalence of Arimoto's bound to that of Dueck and Körner.

key words: Strong converse theorem, discrete memoryless channels, exponent of correct decoding

1. Introduction

In some class of noisy channels the error probability of decoding goes to one as the block length n of transmitted codes tends to infinity at rates above the channel capacity. This is well known as a strong converse theorem for noisy channels. In 1957, Wolfowitz [1] proved the strong converse theorem for discrete of memoryless channels (DMCs). His result is the first result on the strong converse theorem.

In 1973, Arimoto [2] obtained some stronger result on the strong converse theorem for DMCs. He proved that the error probability of decoding goes to one exponentially and derived a lower bound of the exponent function. To prove the above strong converse theorem he introduced an interesting bounding technique based on a symmetrical structure of the set of transmission codes. Using this bounding method and an analytical argument on convex functions developed by Gallager [3], he derived the lower bound.

Subsequently, Dueck and Körner [4] determined the optimal exponent function for the error probability of decoding to go to one. They derived the result by using a combinatorial method base on the type of sequences. Their method is quite different from the method of Arimoto [2]. In their paper, Dueck and Körner [4] stated that their optimal bound can be proved to be equal to the lower bound of Arimoto [2] by

analytical computation. However, after their statement we have found no rigorous proof of the above equality so far in the literature.

In this paper we give a rigorous proof of the equality of the lower bound of Arimoto [2] to that of the optimal bound of Dueck and Körner [4]. To prove the above equality, we need to prove the convex property of the optimal exponent function. We prove this by an *operational meaning* of the optimal exponent function. Contrary to their statement, our arguments of the proof are *not completely analytical*. A dual equivalence of two exponent functions was established by Csiszár and Körner [5] on the exponent functions for the error probability of decoding to go to zero at rates below capacity. Their arguments of the proof of equivalence are *completely analytical*. We compare our arguments to their ones to clarify an essential difference between them.

2. Coding Theorems for Discrete Memoryless Channels

We consider the discrete memoryless channel with the input set \mathcal{X} and the output set \mathcal{Y} . We assume that \mathcal{X} and \mathcal{Y} are finite sets. Let X^n be a random variable taking values in \mathcal{X}^n . Suppose that X^n has a probability distribution on \mathcal{X}^n denoted by $P_{X^n} = \{P_{X^n}(\mathbf{x})\}_{\mathbf{x} \in \mathcal{X}^n}$. Let $Y^n \in \mathcal{Y}^n$ be a random variable obtained as the channel output by connecting X^n to the input of channel. We write a conditional distribution of Y^n on given X^n as $W^n = \{W^n(\mathbf{y}|\mathbf{x})\}_{(\mathbf{x}, \mathbf{y}) \in \mathcal{X}^n \times \mathcal{Y}^n}$. A noisy channel is defined by a sequence of stochastic matrices $\{W^n\}_{n=1}^\infty$. In particular, a stationary discrete memoryless channel is defined by a stochastic matrix with input set \mathcal{X} and output set \mathcal{Y} . We write this stochastic matrix as $W = \{W(y|x)\}_{(x,y) \in \mathcal{X} \times \mathcal{Y}}$.

Information transmission using the above noisy channel is formulated as follows. Let \mathcal{M}_n be a message set to be transmitted through the channel. Set $M_n = |\mathcal{M}_n|$. For given W , a (n, M_n, ε_n) -code is a set of $\{(\mathbf{x}(m), \mathcal{D}(m), m \in \mathcal{M}_n)\}$ that satisfies the following:

- 1) $\mathbf{x}(m) \in \mathcal{X}^n$,
- 2) $\mathcal{D}(m), m \in \mathcal{M}_n$ are disjoint subsets of \mathcal{Y}^n ,
- 3) $\varepsilon_n = \frac{1}{M_n} \sum_{m \in \mathcal{M}_n} W^n((\mathcal{D}(m))^c | \mathbf{x}(m))$,

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*This work was presented in part at the 28rd Symposium on Information Theory and Its Applications, Onna, Okinawa, Japan, Nov. 20-23, 2005.

where $\mathcal{D}(m), m \in \mathcal{M}_n$ are decoding regions of the code and ε_n is the error probability of decoding.

A transmission rate R is achievable if there exists a sequence of (n, M_n, ε_n) -codes, $n = 1, 2, \dots$ such that

$$\limsup_{n \rightarrow \infty} \varepsilon_n = 0, \liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \geq R. \quad (1)$$

Let the supremum of achievable transmission rate R be denoted by $C(W)$, which we call the channel capacity. It is well known that $C(W)$ is given by the following formula:

$$C(W) = \max_{P \in \mathcal{P}(\mathcal{X})} I(P, W), \quad (2)$$

where $\mathcal{P}(\mathcal{X})$ is a set of probability distribution on \mathcal{X} and $I(P, W)$ stands for a mutual information between X and Y when input distribution of X is P .

To examine an asymptotic behavior of ε_n for large n at $R < C(W)$, we define the following quantities. For give $R \geq 0$, the quantity E is achievable error exponent if there exists a sequence of (n, M_n, ε_n) -codes, $n = 1, 2, \dots$ such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \geq R, \liminf_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \log \varepsilon_n \geq E.$$

The supremum of the achievable error exponent E is denoted by $E^*(R|W)$. Several lower and upper bounds of $E^*(R|W)$ have been derived so far. An explicit form of $E^*(R|W)$ is known for large R below $C(W)$. An explicit formula of $E^*(R|W)$ for all R below $C(W)$ has been unknown yet.

3. Strong Converse Theorems for Discrete Memoryless Channels

Wolfowitz [1] first established the strong converse theorem for DMCs by proving that when $R > C(W)$, we have $\lim_{n \rightarrow \infty} \varepsilon_n = 1$. When strong converse theorem holds, we are interested in a rate of convergence for the error probability of decoding to tend to one as $n \rightarrow \infty$ for $R > C(W)$. To examine the above rate of convergence, we define the following quantity. For give $R \geq 0$, the quantity G is achievable exponent if there exists a sequence of (n, M_n, ε_n) -codes, $n = 1, 2, \dots$ such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \geq R, \limsup_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \log (1 - \varepsilon_n) \leq G.$$

The infimum of the achievable exponent G is denoted by $G^*(R|W)$. This quantity has the following property.

Property 1: The function $G^*(R|W)$ is a monotone increasing and convex function of R .

Proof: By definition it is obvious that $G^*(R|W)$ is a monotone increasing function of R . To prove the convexity fix two positive rates R_1, R_2 arbitrary. For each $R_i, i = 1, 2$, we consider the infimum of the achievable exponent function $G^*(R_i|W)$. By the definitions

of $G^*(R_i|W), i = 1, 2$, for each $i = 1, 2$, there exists a sequence of $(n, M_n^{(i)}, \varepsilon_n^{(i)})$ -codes, $n = 1, 2, \dots$, such that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n^{(i)} \geq R_i, \\ \limsup_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \log (1 - \varepsilon_n^{(i)}) \leq G^*(R_i|W).$$

Fix any $\lambda_i, i = 1, 2$ with $\lambda_1 + \lambda_2 = 1$ and set $n_i = \lfloor \lambda_i n \rfloor$, where $\lfloor a \rfloor$ stands for the integer part of a . Set $\nu = n - n_1 - n_2$. It is obvious that $\nu \in \{0, 1, 2\}$.

Next, we consider the code obtained by concatenating $(n_i, M_{n_i}^{(i)}, \varepsilon_{n_i}^{(i)})$ -codes for $i = 1, 2$. If $\nu = 1$ or 2 , we further append $(\nu, 1, 0)$ -code. For the above constructed (n, M_n, ε_n) -code we have

$$M_n = \prod_{i=1,2} M_{n_i}^{(i)}, \quad 1 - \varepsilon_n = \prod_{i=1,2} (1 - \varepsilon_{n_i}^{(i)}).$$

Then, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log M_n \\ = \sum_{i=1,2} \liminf_{n \rightarrow \infty} \frac{n_i}{n} \cdot \frac{1}{n_i} \log M_{n_i}^{(i)} \geq \sum_{i=1,2} \lambda_i R_i, \\ \limsup_{n \rightarrow \infty} \left(-\frac{1}{n} \right) \log (1 - \varepsilon_n) \\ = \sum_{i=1,2} \limsup_{n \rightarrow \infty} \frac{n_i}{n} \cdot \left(-\frac{1}{n_i} \right) \log (1 - \varepsilon_{n_i}^{(i)}) \\ \leq \sum_{i=1,2} \lambda_i G^*(R_i|W).$$

Hence, we have

$$\sum_{i=1,2} \lambda_i G^*(R_i|W) \geq G^* \left(\sum_{i=1,2} \lambda_i R_i \middle| W \right),$$

which implies the convexity of $G^*(R_i|W)$. \square

Arimoto [2] derived a lower bond of $G^*(R|W)$. To state his result we define some functions. For $\delta \in [-1, +\infty)$, define

$$J_\delta(P|W) \triangleq -\log \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P(x) W(y|x)^{\frac{1}{1+\delta}} \right]^{1+\delta}, \\ F_\delta(R, P|W) \triangleq \delta R + J_\delta(P|W), \\ G_\delta(R|W) \triangleq \min_{P \in \mathcal{P}(\mathcal{X})} F_\delta(R, P|W).$$

Furthermore, set

$$G(R|W) \triangleq \max_{-1 \leq \delta \leq 0} G_\delta(R|W) \\ = \max_{-1 \leq \delta \leq 0} \min_{P \in \mathcal{P}(\mathcal{X})} F_\delta(R, P|W) \\ = \max_{-1 \leq \delta \leq 0} \left[-\delta R + \min_{P \in \mathcal{P}(\mathcal{X})} J_\delta(P|W) \right].$$

According to Arimoto [2], the following property holds.

Property 2: The function $G(R|W)$ is a monotone increasing and convex function of R and is positive if and only if $R > C(W)$.

Arimoto [2] proved the following theorem.

Theorem 1: For any $R \geq 0$, $G^*(R|W) \geq G(R|W)$.

Arimoto [2] derived the lower bound $G(R|W)$ of $G^*(R|W)$ by an analytical method. Subsequently, Dueck and Körner [4] determined $G^*(R|W)$ by a combinatorial method quite different from that of Arimoto. To state their result for $P \in \mathcal{P}(\mathcal{X})$ and $R \geq 0$, we define the following function

$$\tilde{F}_\delta^+(R, P|W) \triangleq \min_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \{ [\delta(-R + I(P; V))]^+ + D(V||W|P) \},$$

where $\mathcal{P}(\mathcal{Y}|\mathcal{X})$ is a set of all noisy channels with input \mathcal{X} and output \mathcal{Y} and $[a]^+ = \max\{a, 0\}$. Furthermore, for $R \geq 0$, define

$$\tilde{G}_{-1}^+(R|W) \triangleq \min_{P \in \mathcal{P}(\mathcal{X})} \tilde{F}_{-1}^+(R, P|W),$$

and for $0 \leq R \leq \log |\mathcal{X}|$, define

$$\tilde{G}_{\text{sp}}(R|W) \triangleq \min_{P \in \mathcal{P}(\mathcal{X})} \min_{\substack{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X}): \\ I(P; V) \geq R}} D(V||W|P).$$

The suffix “sp” of the function $\tilde{G}_{\text{sp}}(R|W)$ derives from that it has a form of *the sphere packing exponent function*. Those functions satisfy the following.

Property 3:

- a) The function $\tilde{G}_{\text{sp}}(R|W)$ is monotone increasing for $0 \leq R \leq \log |\mathcal{X}|$ and takes positive value if and only if $R > C(W)$.
- b) For $0 \leq R \leq \log |\mathcal{X}|$, we have

$$\tilde{G}_{-1}^+(R|W) = \tilde{G}_{\text{sp}}(R|W).$$

Furthermore, for $R \geq \log |\mathcal{X}|$, we have

$$\tilde{G}_{-1}^+(R|W) = \tilde{G}_{-1}(R|W).$$

- c) For $R \geq 0$

$$|\tilde{G}_{-1}^+(R|W) - \tilde{G}_{-1}^+(R'|W)| \leq |R - R'|.$$

Proof: Property 3 part a) is obvious. Proof of part c) is found in Dueck and Körner [4]. In this paper we prove the part b). To prove the first inequality, for fixed $P \in \mathcal{P}(\mathcal{X})$, we set

$$\tilde{G}_{\text{sp}}(R, P|W) \triangleq \min_{\substack{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X}): \\ I(P; V) \geq R}} D(V||W|P)$$

$$\begin{aligned} & \hat{F}_{-1}(R, P|W) \\ & \triangleq \min_{\substack{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X}): \\ I(P; V) \leq R}} \{R - I(P; V) + D(V||W|P)\}. \end{aligned}$$

It is obvious that

$$\begin{aligned} & \tilde{F}_{-1}^+(R, P|W) \\ & = \min \{ \tilde{G}_{\text{sp}}(R, P|W), \hat{F}_{-1}(R, P|W) \} \end{aligned} \quad (3)$$

$$\tilde{G}_{\text{sp}}(R|W) = \min_{P \in \mathcal{P}(\mathcal{X})} \tilde{G}_{\text{sp}}(R, P|W). \quad (4)$$

Since $-I(P; V) + D(V||W|P)$ is a linear function of V , the minimum is attained by some V satisfying $I(P; V) = R$. Then, by (3), we have

$$\tilde{F}_{-1}^+(R, P|W) = \tilde{G}_{\text{sp}}(R, P|W).$$

From the above equality and (4), we obtain the first equality. The second equality is obvious since $R - I(P; V) \geq 0$ when $R \geq \log |\mathcal{X}|$. \square

Dueck and Körner [4] proved the following.

Theorem 2: For any $R > 0$,

$$\tilde{G}_{-1}^+(R|W) = G^*(R|W).$$

Although the lower bound derived by Arimoto [2] is a form quite different from the optimal exponent determined by Dueck and Körner [4], the former coincides with the latter, i.e., the following theorem holds.

Theorem 3: For any $R \geq 0$,

$$\tilde{G}_{-1}^+(R|W) = G(R|W),$$

or equivalent to

$$\begin{aligned} & \max_{-1 \leq \delta \leq 0} \min_{P \in \mathcal{P}(\mathcal{X})} \left\{ -\delta R \right. \\ & \quad \left. - \log \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P(x) W(y|x)^{\frac{1}{1+\delta}} \right]^{1+\delta} \right\} \\ & = \min_{P \in \mathcal{P}(\mathcal{X})} \min_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \{ [R - I(P; V)]^+ + D(V||W|P) \}. \end{aligned}$$

The result of Theorem 3 is stated in Csiszár and Körner [5] without proof. Dueck and Körner [4] stated that the equivalence between their bound and that of Arimoto [2] can be proved by an analytical computation. In the next section we give a rigorous proof of the above theorem. Contrary to their statement, our proof is *not completely analytical*.

4. Proof of Theorem 3

In this section we prove Theorem 3. The following is a key lemma for the proof.

Lemma 1: The function $\tilde{G}_{-1}^+(R|W)$ is a monotone increasing and convex function of $R \geq 0$.

Proof: The results follows from the convexity of $G^*(R|W)$ and Theorem 2. \square

Remark 1: We first tried to prove Lemma 1 by an analytical computation but could not succeed proving this lemma via this approach. According to [6], for each fixed $P \in \mathcal{P}(\mathcal{X})$, $\tilde{F}_{-1}^+(R, P|W)$ is a convex function of $R \geq 0$. However, this does not imply the convexity of $\tilde{G}_{-1}^+(R|W)$ with respect to $R \geq 0$.

Next, for $R \geq 0$, we set

$$\begin{aligned}\tilde{F}_\delta(R, P|W) &\triangleq \min_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \{ \delta[I(P; V) - R] \\ &\quad + D(V||W|P) \}, \\ \tilde{G}_\delta(R|W) &\triangleq \min_{P \in \mathcal{P}(\mathcal{X})} \tilde{F}_\delta(R, P|W).\end{aligned}$$

Then, we have the following two lemmas.

Lemma 2: For any $R \geq 0$,

$$\tilde{G}_{-1}^+(R|W) = \max_{-1 \leq \delta \leq 0} \tilde{G}_\delta(R|W).$$

Lemma 3: For any $R \geq 0$, $-1 \leq \delta \leq 0$ and any $P \in \mathcal{P}(\mathcal{X})$, we have

$$\tilde{F}_\delta(R, P|W) \geq F_\delta(R, P|W).$$

Furthermore, for any $R \geq 0$ and $-1 \leq \delta \leq 0$,

$$\tilde{G}_\delta(R|W) = G_\delta(R|W).$$

It is obvious that Theorem 3 immediately follows from Lemmas 2 and 3. Those two lemmas can be proved by *analytical computations*. In the following we prove Lemma 2. The proof of Lemma 3 is omitted here. For the detail see Oohama [7].

Proof of Lemma 2: From its formula, it is obvious that

$$\tilde{G}_{-1}^+(R|W) \geq \max_{-1 \leq \delta \leq 0} \tilde{G}_\delta(R|W).$$

In particular, from Property 3 part b), the equality holds for $R \geq \log |\mathcal{X}|$. Then, again by Property 3 part b), it suffices to prove that for $0 \leq R \leq \log |\mathcal{X}|$, there exists $-1 \leq \delta \leq 0$ such that

$$\tilde{G}_{\text{sp}}(R|W) = \tilde{G}_\delta(R|W).$$

For $-1 \leq \delta \leq 0$, we set

$$\begin{aligned}K_\delta(W) &\triangleq \max_{P \in \mathcal{P}(\mathcal{X})} \max_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} \{ -\delta I(P; V) - D(V||W|P) \}.\end{aligned}$$

Then, by the definition of $\tilde{G}_\delta(R|W)$, we have the following.

$$\tilde{G}_\delta(R|W) = -\delta R - K_\delta(W).$$

Next, observe that by Property 3 part b) and Lemma 1, $\tilde{G}_{\text{sp}}(R|W)$ is a monotone increasing and convex function of R . By this property and Property 3 part c), for any $0 \leq R \leq \log |\mathcal{X}|$, there exists $-1 \leq \delta \leq 0$ such that for any $0 \leq R' \leq \log |\mathcal{X}|$, we have

$$\tilde{G}_{\text{sp}}(R'|W) \geq \tilde{G}_{\text{sp}}(R|W) - \delta(R' - R).$$

Let $(P, V) \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ be a joint distribution that attains $\tilde{G}(R|W)$. For any $(P', V') \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ set $R' = I(P'; V')$. Then, we have the following chain of inequalities:

$$\begin{aligned}\delta I(P'; V') - D(V'||W|P') \\ \leq -\delta R' - \tilde{G}_{\text{sp}}(R'|W) \leq -\delta R - \tilde{G}_{\text{sp}}(R|W) \\ \leq -\delta I(P; V) - D(V||W|P).\end{aligned}$$

The above inequality implies that

$$\begin{aligned}K_\delta(W) &= -\delta I(P; V) - D(V||W|P) \\ &= -\delta R - \tilde{G}_{\text{sp}}(R|W).\end{aligned}$$

This completes the proof. \square

5. Comparison with the Proof of the Dual Result

Theorem 3 has some duality with a result stated in Csiszár and Körner [5]. To describe their result we define

$$\begin{aligned}E_\delta(R|W) &\triangleq \max_{P \in \mathcal{P}(\mathcal{X})} F_\delta(R, P|W), \\ E(R|W) &\triangleq \max_{\delta \geq 0} E_\delta(R|W) \\ &= \max_{\delta \geq 0} \max_{P \in \mathcal{P}(\mathcal{X})} F_\delta(R, P|W) \\ &= \max_{\delta \geq 0} \left[-\delta R + \max_{P \in \mathcal{P}(\mathcal{X})} J_\delta(P|W) \right].\end{aligned}$$

An explicit lower bound of $E^*(R|W)$ is first derived by Gallager [8]. He showed that the function $\max_{0 \leq \delta \leq 1} E_\delta(R|W)$ serves as an lower bound of $E^*(R|W)$. Next, we set

$$C_0(W) \triangleq \max_{P \in \mathcal{P}(\mathcal{X})} \min_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X})} I(P; V)$$

According to Shannon, Gallager and Berlekamp [9], $C_0(W)$ has the following formula:

$$C_0(W) = - \min_{P \in \mathcal{P}(\mathcal{X})} \max_{y \in \mathcal{Y}} \log \sum_{x \in \mathcal{X}: W(y|x) > 0} P(x).$$

For $R \geq C_0(W)$, define

$$\tilde{E}_{\text{sp}}(R|W) \triangleq \max_{P \in \mathcal{P}(\mathcal{X})} \min_{\substack{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X}): \\ I(P; V) \leq R}} D(V||W|P).$$

According to Csiszár and Körner [5], $\tilde{E}_{\text{sp}}(R|W)$ serves as an upper bound of $E^*(R|W)$ and matches it for large R below $C(W)$. Csiszár and Körner [5] obtained the following result.

Theorem 4 (Csiszár and Körner [5]): For any $R \geq C_0(W)$,

$$E(R|W) = \tilde{E}_{\text{sp}}(R|W),$$

or equivalent to

$$\begin{aligned} & \max_{\delta \geq 0} \max_{P \in \mathcal{P}(\mathcal{X})} \left\{ -\delta R \right. \\ & \quad \left. - \log \sum_{y \in \mathcal{Y}} \left[\sum_{x \in \mathcal{X}} P(x) W(y|x)^{\frac{1}{1+\delta}} \right]^{1+\delta} \right\} \\ &= \max_{P \in \mathcal{P}(\mathcal{X})} \min_{V \in \mathcal{P}(\mathcal{Y}|\mathcal{X}): I(P;V) \leq R} D(V||W|P). \end{aligned}$$

In the following we outline the arguments of the proof of the above theorem and compare them with those of the proof of Theorem 3.

By an analytical computation we have the following lemma.

Lemma 4: The function $\tilde{E}_{\text{sp}}(R|W)$ is a monotone decreasing and convex function of $R \geq C_0(W)$ and is positive if and only if $C_0(W) \leq R < C(W)$.

Next, for $R \geq 0$, we define

$$\tilde{E}_{\delta}(R|W) = \min_{P \in \mathcal{P}(\mathcal{X})} \tilde{F}_{\delta}(R, P|W).$$

Then, we have the following two lemmas

Lemma 5: For any $R \geq C_0(W)$,

$$\tilde{E}_{\text{sp}}(R|W) = \max_{\delta \geq 0} \tilde{E}_{\delta}(R|W).$$

Lemma 6: For any $R \geq 0$, $\delta \geq 0$ and any $P \in \mathcal{P}(\mathcal{X})$, we have

$$\tilde{F}_{\delta}(R, P|W) \geq F_{\delta}(R, P|W).$$

Furthermore, for any $R \geq 0$ and $\delta \geq 0$,

$$\tilde{E}_{\delta}(R|W) = E_{\delta}(R|W).$$

It is obvious that Theorem 4 immediately follows from Lemmas 5 and 6. We prove Lemmas 5 and 6 in manners quite similar to those of the proofs of Lemmas 2 and 3, respectively. We omit the details of the proofs.

We compare the arguments of the proof of Theorem 3 with those of the proof of Theorem 4. An essential difference between them is in the proof of the convexity of exponent functions. We can prove the convexity of $\tilde{E}_{\text{sp}}(R|W)$ with an analytical method. On the other hand, the convexity $\tilde{G}_{-1}^{+}(R|W)$ follows from $G^{*}(R|W) = \tilde{G}_{-1}^{+}(R|W)$ and the convexity of $G^{*}(R|W)$. The proof of the convexity of $G^{*}(R|W)$ is based on an operational meaning of the optimal exponent function of $1 - \varepsilon_n$. We first tried an analytical proof of the convexity $\tilde{G}_{-1}^{+}(R|W)$ but could not have succeeded in it. The difference of arguments is summarized in TABLE 1.

$R > C(W)$	$R < C(W)$
$G^{*}(R W) = \tilde{G}_{-1}^{+}(R W)$ (Theorem 2)	$E^{*}(R W) \leq \tilde{E}_{\text{sp}}(R W)$ (Open Problem)
Operational Meaning \Downarrow Convexity of $G^{*}(R W)$ (Property 1)	Convexity of $E^{*}(R W)$?
Theorem 2 and Property 1 \Downarrow Convexity of $\tilde{G}_{-1}^{+}(R W)$ (Lemma 1) \Downarrow Lemma 2	Analytical Computation \Downarrow Convexity of $\tilde{E}_{\text{sp}}(R W)$ (Lemma 4) \Downarrow Lemma 5
Lemmas 2 and 3 \Downarrow $G(R W) = \tilde{G}_{-1}^{+}(R W)$ (Theorem 3)	Lemmas 5 and 6 \Downarrow $E(R W) = \tilde{E}_{\text{sp}}(R W)$ (Theorem 4)

Table 1 Difference between the arguments of the proof of Theorem 3 and those of the proof of Theorem 4.

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